15 More Remarks on Solutions to Initial Boundary Value Problems

Example A: Robin b.c.s
Consider

$$\begin{cases} u_t = u_{xx} & 0 < x < 1\ ,\ t > 0 \\ u(x,0) = f(x) & 0 < x < 1 \\ u(0,t) = 0\ ,\ u_x(1,t) + au(1,t) = 0 & t > 0,\ a > 0 \end{cases}$$

Separating variables, $u = T(t)\phi(x)$, we obtain the EVP

$$\begin{cases} \frac{d^2\phi}{dx^2} + \lambda\phi = 0 & 0 < x < 1 \\ \phi(0) = 0\ ,\ \frac{d\phi}{dx}(1) + a\phi(1) = 0 \end{cases}$$

Again, any eigenvalue of this problem must be real. What about $\lambda = 0$? In this case, $d^2\phi/dx^2 = 0$, $\phi(0) = 0$, so $\phi(x) = Ax$. The second b.c. implies $\phi'(1) + a\phi(1) = A(1+a) = 0$, so $\lambda = 0$ is an eigenvalue if and only if $a = -1$. But we stated in the problem statement that $a > 0$ so $\lambda = 0$ is not an eigenvalue for this problem.

If $\{\lambda, \phi\}$ is any eigenvalue-eigenfunction pair, multiply the equation by ϕ and integrate. Then, after an integration-by-parts,

$$\lambda \int_0^1 (\phi(x))^2\, dx = \int_0^1 (\phi^{'}(x))^2\, dx + a\phi(1)^2 \ ,$$

so λ must be positive.

Now $\phi(x) = A\cos(\sqrt{\lambda}x) + B\sin(\sqrt{\lambda}x)$, but because $\phi(0) = 0$, then $A = 0, B \neq 0$, B is arbitrary, so $\phi(x) = \sin(\sqrt{\lambda}x)$ and the b.c. at $x = 1$ gives

$$\sqrt{\lambda}\cos(\sqrt{\lambda}) + a\sin(\sqrt{\lambda}) = 0 \rightarrow \tan(\sqrt{\lambda}) = -\frac{\sqrt{\lambda}}{a}\ .$$

That is, Robin b.c.s generally leads to transcendental equations for determining the eigenvalues.

What if, for example, that $a < 0$, and for notational convenience, let $r = \sqrt{\lambda}$. In figure 1 we have plotted $tan(r)$ and $-r/a$ versus r, so that we can see from the graphs that there are an infinite number of $r_n > 0$, and hence for an order sequence of eigenvalues $\lambda_n = r_n^2$. For example, if

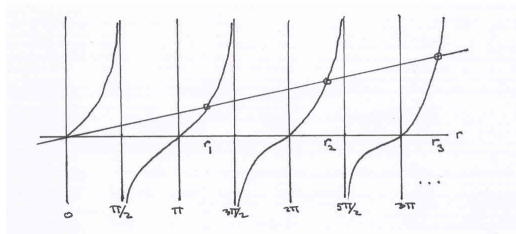

Figure 1: Shows the start of an infinite ordered set of positive eigenvalues for example A, with $a = -1$.

$a = -1$, then $r_1 \cong 4.493$, $r_2 \cong 7.725$, $r_3 \cong 10.9$, so that $\lambda_1 \cong 20.19$, $\lambda_2 \cong 59.68$, $\lambda_3 \cong 118.9$. In fact, for large n, $r_n \cong (n - 1/2)\pi$. (In this case we also have $\lambda_0 = 0$ as an eigenvalue.) Given that we stated $a > 0$, we have a similar situation, but the straight line in figure 1 would have a negative slope.

Exercise: For example A, show there are no negative eigenvalues if $a \geq 0$, or if $a < -1$. Show there is a unique negative eigenvalue if $a \in (-1, 0)$. From a physical standpoint why is this bad, and therefore the proper condition is to have $a > 0$?

Example B: Consider a bit more general case of Robin b.c.s on both ends, namely

$$\begin{cases} u_t = u_{xx} & 0 < x < 1 \ , \ t > 0 \\ u(x,0) = f(x) & 0 < x < 1 \\ -u_x(0,t) + \alpha u(0,t) = 0 \ , \ u_x(1,t) + \beta u(1,t) = 0 & t > 0, \ \alpha, \beta > 0 \end{cases}$$

By separation of variables, $u(x,t) = T(t)\phi(x)$, ϕ satisfies

$$\begin{cases} \frac{d^2\phi}{dx^2} + \lambda\phi = 0 & 0 < x < 1 \\ \frac{d\phi}{dx}(0) - \alpha\phi(0) = 0 \ , \ \frac{d\phi}{dx}(1) + \beta\phi(1) = 0 \ . \end{cases} \tag{1}$$

Exercise: If $\alpha, \beta > 0$, show that any real eigenvalues λ must be non-negative.

Is $\lambda = 0$ an eigenvalue for the problem? If $\lambda = 0$, then $\phi''(x) = 0 \quad \rightarrow \phi = Ax + B$, so applying the b.c.s gives $A - \alpha B = 0 = A + \beta(A + B)$, or in matrix form

$$\begin{bmatrix} 1 & -\alpha \\ 1+\beta & \beta \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} ,$$

which has a nontrivial solution if and only if the determinent of the matrix is zero, namely $\alpha + \beta + \alpha\beta = 0$. If $\alpha, \beta > 0$ this is not possible. For $\lambda > 0$, a solution to (1) is $\phi(x) = A\cos(\sqrt{\lambda}x) + B\sin(\sqrt{\lambda}x)$, so applying the b.c.s gives

$$\begin{bmatrix} -\alpha & \sqrt{\lambda} \\ -\sqrt{\lambda}\sin(\sqrt{\lambda}) + \beta\cos(\sqrt{\lambda}) & \sqrt{\lambda}\cos(\sqrt{\lambda}) + \beta\sin(\sqrt{\lambda}) \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} ,$$

so we have a nontrivial solution if and only if

$$-(\alpha+\beta)\sqrt{\lambda}\cos(\sqrt{\lambda}) + (\lambda - \alpha\beta)\sin(\sqrt{\lambda}) = 0 \, .$$

Let $r := \sqrt{\lambda}$; then we can write this expression as

$$tan(r) = \frac{(\alpha+\beta)r}{r^2 - \alpha\beta} \quad . \tag{2}$$

Exercises
For this exercise let us forgo the physical constraint that α and β should be positive.

1. If $\beta = -\alpha$, note that $r = 0$ is a root of (2), but that $\lambda = 0$ is not an eigenvalue in this case.

2. If $\alpha+\beta > 0$ and $\alpha < 0 < \beta$, show there is an infinite number of positive eigenvalues with the first one being in $(0, \pi/2)$; show it is possible to have $\lambda = 0$ an eigenvalue; If, however, $0 < \alpha,\ \beta$, then $\lambda = 0$ is not an eigenvalue, but there is still an infinite number of positive eigenvalues. Thus, this should be the only physically meaningful case for Robin b.c.s on both ends.

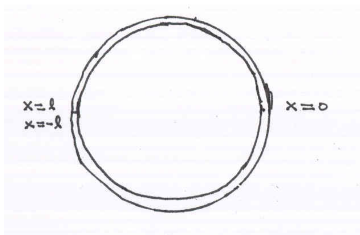

Figure 2: Sketch of a thin ring showing an imaginary cut to define the domain for example problem C.

3. What about negative eigenvalues? Let $\lambda = -k^2,\ k > 0$. Analyze the analogue to (2).

Example C: Heat conduction in a thin circular ring
Consider

$$\begin{cases} u_t = Du_{xx} & -l < x < l\ ,\ t > 0 \\ u(x,0) = f(x) & -l < x < l \end{cases}$$

Here we imagine a very thin, insulated ring (cross-sectionally isothermal and no heat loss/gain through sides) that is of length $2l$. (See figure 2.) Imagine an arbitrary cut and ring uncurled and laid out straight (visually). Then our spatial domain is $-l < x < l$. But the ring is not actually cut, so we must have continuity of temperature and its gradient across the imaginary cut at $x = l$. Hence, the (periodic) b.c.s are

$$\begin{aligned} u(-l,t) &= u(l,t) \qquad t > 0 \\ u_x(-l,t) &= u_x(l,t) \end{aligned}$$

Remark: These periodic boundary conditions represent a special case of **mixed** boundary conditions, where there is a relationship between u (or its derivatives) at both endpoints.

Employing separation of variables, $u(x,t) = T(t)\phi(x)$, we obtain the usual

$\frac{dT}{dt} = -\lambda DT \quad \rightarrow \quad T(t) = e^{-\lambda Dt}$, along with the EVP

$$\begin{cases} \frac{d^2\phi}{dx^2} + \lambda\phi = 0 & -l < x < l \\ \\ \phi(-l) = \phi(l) \\ \\ \frac{d\phi}{dx}(-l) = \frac{d\phi}{dx}(l) \end{cases}$$

Therefore, making the assumption the eigenvalue $\lambda \geq 0$ (which has to be checked later), consider first the case $\lambda > 0$. Then $\phi(x) = A\cos(\sqrt{\lambda}x) + B\sin(\sqrt{\lambda}x)$. Now

$$A\cos(\sqrt{\lambda}l) + B\sin(\sqrt{\lambda}l) = \phi(l) = \phi(-l) =$$
$$A\cos(\sqrt{\lambda}l) - B\sin(\sqrt{\lambda}l) ,$$

which implies $2B\sin(\sqrt{\lambda}l) = 0$. So either $B = 0$, or $\sin(\sqrt{\lambda}l) = 0$, the latter implying $\lambda = \lambda_n = (n\pi/l)^2$, $n = 1, 2, \ldots$. The second boundary condition gives

$$\sqrt{\lambda}\{-A\sin(\sqrt{\lambda}l) + B\cos(\sqrt{\lambda}l)\} = \frac{d\phi}{dx}(l) = \frac{d\phi}{dx}(-l) =$$
$$\sqrt{\lambda}\{A\sin(\sqrt{\lambda}l) + B\cos(\sqrt{\lambda}l)\} ,$$

so that $2A\sqrt{\lambda}\sin(\sqrt{\lambda}l) = 0$. Thus, either $A = 0$, or $\sin(\sqrt{\lambda}l) = 0$. That is, if $\sin(\sqrt{\lambda}l) \neq 0$, then $A = B = 0$; therefore, $\lambda = \lambda_n = (n\pi/l)^2$, $n = 1, 2, \ldots$.

What about the case where $\lambda = 0$? For that, $\phi(x) = Ax + B$, and the first b.c. gives

$$Al + B = \phi(l) = \phi(-l) = -Al + B \quad \rightarrow \quad 2Al = 0 \quad \rightarrow \quad A = 0 .$$

So $\phi = B$ automatically satisfies the second b.c., and with B being arbitrary, $\lambda = 0$ is an eigenvalue for the problem. Its associated eigenfunction is just a constant. To summarize,

$$\phi_n = \begin{cases} a_0/2 & n = 0 \\ \\ a_n\cos(\frac{n\pi x}{l}) + b_n\sin(\frac{n\pi x}{l}) & n \geq 1 , \end{cases}$$

along with $T(t) = T_n(t) = e^{-n^2\pi^2 Dt/l^2}$, we have

$$u(x,t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} e^{-n^2\pi^2 Dt/l^2}\{a_n\cos(\frac{n\pi x}{l}) + b_n\sin(\frac{n\pi x}{l})\} .$$

To find the Fourier coefficients, let $t \to 0$ and consider

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \{a_n \cos(\frac{n\pi x}{l}) + b_n \sin(\frac{n\pi x}{l})\} \ .$$

In this example, where the domain is defined over the whole symmetric interval $[-l, l]$, f is arbitrary in that, other than some smoothness condition, f is neither even or odd, so we are led to consideration of the *full* Fourier series for f.

As we develop our ideas here about classical Fourier series (trig. eigenfunction series), keep in mind the examples from the last section:

Example 1:

$$\begin{cases} u_t = u_{xx} & 0 < x < 1 \ , \ t > 0 \quad \text{here } D = 1, l = 1 \\ u(x,0) = R & 0 < x < 1 \\ u(0,t) = 0 \ , \ u(1,t) = 0 & t > 0 \ , \end{cases}$$

so that

$$u(x,t) = \frac{4R}{\pi} \sum_{k=1}^{\infty} \frac{e^{-(2k-1)^2\pi^2 t}}{2k-1} \sin((2k-1)\pi x) \ .$$

Example 2:

$$\begin{cases} u_t = u_{xx} & 0 < x < 1 \ , \ t > 0 \quad \text{before we had } l = 2 \\ u(x,0) = x & 0 < x < 1 \\ u(0,t) = 0 \ , \ u(1,t) = 0 & t > 0 \ , \end{cases}$$

so that

$$u(x,t) = \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1} e^{-n^2\pi^2 t}}{n} \sin(n\pi x) \ .$$

Example 3:

$$\begin{cases} u_t = u_{xx} & 0 < x < 1 \ , \ t > 0 \\ u(x,0) = x & 0 < x < 1 \\ u_x(0,t) = 0 \ , \ u_x(1,t) = 0 & t > 0 \ , \end{cases}$$

so that

$$u(x,t) = \frac{1}{2} - \frac{4}{\pi^2} \sum_{k=1}^{\infty} \frac{e^{-(2k-1)^2\pi^2 t}}{(2k-1)^2} \cos((2k-1)\pi x) \ .$$

Remark: In example 1, let $t \to 0$ and set $x = 1/2$:

$$R = \frac{4R}{\pi} \sum_{k=1}^{\infty} \frac{\sin((k-1/2)\pi x)}{2k-1} = \frac{4R}{\pi} \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{2k-1} \ , \qquad \text{or}$$

$$\frac{\pi}{4} = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{2k-1} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots \ .$$

In example 3, set $t = 0$ and $x = 0$; then

$$0 = \frac{1}{2} - \frac{4}{\pi^2} \sum_{k=1}^{\infty} \frac{1}{(2k-1)^2} \ , \quad \text{or}$$

$$\frac{\pi^2}{8} = \sum_{k=1}^{\infty} \frac{1}{(2k-1)^2} = 1 + \frac{1}{9} + \frac{1}{25} + \dots \ .$$

In calculus you generally only study conditions under which infinite series converge (or diverge). But there are tables of infinite series that give values they converge to. Most often it is through calculating Fourier series for various functions and evaluating them at a specific value of x that give one the convergent value of a series.

Remark: From example 1,

$$u(x,t) = \frac{4R}{\pi} \{ e^{-\pi^2 t} \sin(\pi x) + \frac{1}{3} e^{-9\pi^2 t} \sin(3\pi x) + \dots \} \ .$$

Consider the relative sizes of the first and second terms:

$$\frac{|\text{second term}|}{|\text{first term}|} = \frac{1}{3} e^{-8\pi^2 t} |\frac{sin(3\pi x)}{\sin(\pi x)}| \leq e^{-8\pi^2 t} \leq e^{-8} < 0.00034 \ ,$$

where we have used the fact that $|\sin(nx)|n \leq |\sin(x)|$ and we assume $t > 1/\pi^2 \approx 0.1013$. That is, as soon as t is away from 0 by $1/\pi^2$, the second term is on the order of 10^{-4} times the size of the first term. Put another way, for $t > 1/\pi^2$,

$$u(x,t) \approx \frac{4R}{\pi} e^{-\pi^2 t} \sin(\pi x)$$

is a pretty good approximation for $u(x,t)$. The difference in successive terms is even magnified more the further one goes out in the series, so individual terms die very quickly away as soon as we leave $t = 0$.

Let us look at the situation in a slightly more general context, that is, let $u(x,0) = f(x)$ for any absolutely integrable function (same heat equation with homogeneous Dirichlet b.c.s on both ends):

$$u(x,t) = \sum_{n=1}^{\infty} b_n e^{-n^2\pi^2 t} \sin(n\pi x) = \sum_{n=1}^{\infty} u_n(x,t)$$

$$b_n = 2\int_0^1 f(x)\sin(n\pi x)dx$$

Now $|b_n| \le 2\int_0^1 |f(x)||\sin(n\pi x)|dx \le 2\int_0^1 |f(x)|dx < \infty$. Since b_n is bounded by a constant independent of n, just write $|b_n| \le B$. (Coefficients are uniformly bounded.) For each "mode" , u_n,

$$|u_n(x,t)| \le Be^{-n^2\pi^2 t} \le Be^{-n\pi^2 t} = B(e^{-\pi^2 t})^n \ .$$

Recall the geometric series

$$\sum_{n=1}^{\infty} r^n = \frac{r}{1-r} \quad \text{for } |r| < 1 \ .$$

Therefore, for $t > 0$ fixed, $r \doteq e^{-\pi^2 t}$,

$$\left|\sum_{n=1}^{\infty} b_n e^{-n^2\pi^2 t}\sin(n\pi x)\right| \le \sum_{n=1}^{\infty} |b_n|(e^{-\pi^2 t})^n \le B\sum_{n=1}^{\infty} r^n = \frac{Be^{-\pi^2 t}}{1-e^{-\pi^2 t}} \ .$$

What have we learned from the above Fourier series solution of the heat equation?

1. $u(x,t) \ \to \ 0$ *exponentially* fast ($u \equiv 0$ is the problem's steady state solution); so the rod cools very quickly to its steady state temperature imposed on the ends;

2. The first term in the eigenfunction series, $b_1 e^{-\pi^2 t}\sin(\pi x)$, that is, the one with the smallest eigenvalue, determines the decay rate of u.

Remark: Consider

$$\text{Example 4:} \quad \begin{cases} u_t = u_{xx} & 0 < x < 1 \ , \ t > 0 \\ u(x,0) = x(1-x) & 0 < x < 1 \\ u(0,t) = 0 = u(1,t) & t > 0 \ . \end{cases}$$

Since $\int_0^1 x(1-x)\sin(n\pi x)dx = -2\frac{[(-1)^n-1]}{(n\pi)^3}$, we have

$$u(x,t) = \frac{8}{\pi^3}\sum_{n=1,3,5,..}\frac{e^{-n^2\pi^2 t}}{n^3}\sin(n\pi x) = \frac{8}{\pi^3}\sum_{k=1}^{\infty}\frac{e^{-(2k-1)^2\pi^2 t}}{(2k-1)^3}\sin((2k-1)\pi x)$$

Exercise: Verify this solution.

With the same b.c.s as in example 2, when $f(x) = x$ the coefficients b_n decay like $1/n$ as $n \to \infty$. When $f(x) = x(1-x)$, the coefficients decay like $1/n^3$, that is, much faster, as $n \to \infty$. Why?

This requires us to analyze what is going on with Fourier series. To get a handle on the eigenfunction series for solutions to heat equations (or the vibrating string equations), we must analyze the series situation for the initial data.

The type of series depends on the EVP, specifically the imposed boundary conditions at $x = 0, l$. In determining the Fourier coefficients we have set the function to the series, multiplied by an arbitrary eigenfunction, and integrated both sides. This assumes a number of properties we have not addressed:

- that we have orthogonality of the eigenfunctions on the given interval;
- that the series converges so that the series even makes sense;
- that we can interchange integration and the infinite series.

We will discuss point one when we get to the section on Sturm-Liouville EVPs. We will discuss the second point in the next Section and afterward discuss the third point. To do this we should have a class of functions in mind. For our purposes, and most physical situations, the set of piecewise smooth functions introduced in the next section should be sufficiently general.